

A Farey tail for attractor black holes

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ABSTRACT: The microstates of 4d BPS black holes in IIA string theory compactified on a Calabi-Yau manifold are counted by a (generalized) elliptic genus of a (0,4) conformal field theory. By exploiting a spectral flow that relates states with different charges, and using the Rademacher formula, we find that the elliptic genus has an exact asymptotic expansion in terms of semi-classical saddle-points of the dual supergravity theory. This generalizes the known "Black Hole Farey Tail" of [25] to the case of attractor black holes.

KEYWORDS: Black Holes in String Theory, AdS-CFT Correspondence.

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1. Introduction

One of the main successes of string theory has been the microscopic explanation of black hole entropy. The microstates for extremal BPS black holes are well understood in theories with 16 or more supercharges. This includes the original D1-D5-P system in type IIB theory on $K3 \times S^1$ for which the microstates are represented by the elliptic genus of a (4,4) CFT with target space given by a symmetric product of $K3$ [2, 3]. The elliptic genus for this target space can be explicitly computed, leading to a concrete and exact expression for the number of BPS-states.

The D1-D5-P system has a well understood dual description in terms of type IIB theory on $K3 \times AdS_3 \times S^3$. A rather remarkable result, known as the Rademacher series, is that the elliptic genus has an exact asymptotic expansion, which has a natural interpretation as a sum over semi-classical contributions of saddle-point configurations of the dual supergravity theory. This exact asymptotic expansion, together with its semi-classical interpretation, has been coined the Black Hole Farey Tail [25]. Although the Farey tail was first introduced in the context of the D1-D5 system, it applies to any system that has a

microscopic description in terms of a (decoupled) 2d conformal field theory and has a dual description as a string/supergravity theory on a spacetime that contains an asymptotically AdS_3 .

The aim of this paper is to apply the Rademacher formula to black holes in theories with eight supercharges and in this way extend the Farey Tail to $N=2$ (or attractor) black holes. Specifically, we consider M-theory compactified on a Calabi-Yau three-fold X and study the supersymmetric bound states of wrapped M5-branes with M2-branes. These states correspond to extremal four dimensional black holes after further reduction on a circle. For this situation a microscopic description was proposed quite a while ago by Maldacena, Strominger and Witten (MSW) [4], who showed that the black hole microstates are represented by the supersymmetric ground states of a (0,4) conformal field theory. These states are counted by an appropriately defined elliptic genus of the (0,4) CFT.

The interest in attractor black holes has been revived in recent years due to the connection with topological string theory discovered in [5, 6] and subsequently studied by many different authors. In particular, it was conjectured by Ooguri, Strominger and Vafa (OSV) in [6] that the mixed partition function of 4d BPS black holes is equal to the absolute value squared of the topological string partition function. Earlier, in a separate development, a different connection between BPS states and topological strings was discovered by Gopakumar and Vafa (GV) [7], who showed that topological string theory computes the number of five-dimensional BPS-invariants of wrapped M2 branes in M-theory on a Calabi-Yau. The GV-result differs from the OSV-conjecture in the sense that the topological string coupling constant appears in an S-dual way. Recently, this fact and the OSV conjecture have been considerably clarified in the work of Gaiotto, Strominger, and Yin [8]. These authors used the CFT approach of MSW to show that the elliptic genus of the (0,4) CFT has a low temperature expansion which (approximately) looks like the square of the GV-partition function. The OSV conjecture then follows from the modular invariance of the elliptic genus, which at the same time naturally explains the different appearances of the coupling constant.

In this paper we will show that elliptic genus of the (0,4) SCFT can be written as a Rademacher series (or Farey tail expansion) similar to that of the previously studied (4,4) case.¹ An important property of the SCFT is the presence of a spectral flow that relates states with different charges, and implies that the elliptic genus can be expanded in terms of theta functions. These theta functions signal the presence of a set of chiral scalars in the SCFT, while from a spacetime point of view their appearance naturally follows from the Chern-Simons term in the effective action. We find that the Farey tail expansion contains subleading contributions to each saddle point that can be interpreted as being due to a virtual cloud of BPS-particles (actually, wrapped M2-branes) that are "light" enough so that by themselves they do not form a black hole. The degeneracies of these particles are, in the large central charge limit, given in terms of the Gopakumar-Vafa invariants. In this way we see that the results of [8] naturally fit in and to a certain extent follow from our Attractor Farey Tail.

¹Some related results were obtained independently in [9–11].

The outline of the paper is as follows: in section 2 we review the bound states of wrapped M5 and M2 branes in M-theory on a Calabi-Yau three-fold and explain the emergence of the spectral flow. We then discuss the decoupling limit and the near horizon geometry and describe the dimensionally reduced effective action on AdS_3 . In section 3 we turn to the M5 brane world volume theory and its reduction to the (0,4) SCFT. Here we also define the generalized elliptic genus which will give us the graded black hole degeneracies. In section 4 we recapitulate the Rademacher formula and Farey Tail expansion, and subsequently apply it to the elliptic genus defined earlier. In section 5 we interpret our result from the dual supergravity perspective and discuss its relation to the OSV conjecture. Finally in section 6 we present our conclusions and raise some open questions.

2. Wrapped M-branes and the near horizon limit

To establish the notation, in this section we describe the BPS bound states of wrapped M5 and M2 branes in M-theory on a Calabi-Yau from a spacetime point of view. We will derive a spectral flow symmetry relating states with different M2 and M5 brane charge, first from an eleven-dimensional perspective and subsequently in terms of the effective three dimensional supergravity that appears in the near horizon limit.

2.1 Wrapped branes on Calabi-Yau and the spectral flow

Consider an M5-brane (or possibly multiple M5-branes) wrapping a 4-cycle $\mathcal{P} = p^a D_a$ in the Calabi-Yau three-fold X . Here $\{D_a\}$ is a basis of integral 4-cycles $H_4(X, \mathbb{Z})$ in X . In order for this 5-brane to be supersymmetric, the 4-cycle \mathcal{P} has to be realized as a positive divisor. The wrapped M5-brane thus reduces to a string in the remaining five dimensions. In addition there are five-dimensional particles corresponding to M2-branes wrapping a two-cycle $\Sigma = q_a \Sigma^a$, where Σ^a is a basis of $H_2(X, \mathbb{Z})$ dual to $\{D_a\}$, i.e. $\Sigma^a \cap D_b = \delta_b^a$. These particles carry charges q_a under the U(1) gauge fields A_a which arise from the reduction of the M-theory 3-form C

$$C = \sum_a A^a \wedge \alpha_a, \tag{2.1}$$

where $\alpha_a \in H^2(X, \mathbb{Z})$ is a basis of harmonic 2-forms Poincaré dual to the four-cycles D_a . Such an ensemble of strings and particles can form a BPS bound state which leaves four out of the eight supersymmetries unbroken.

Eventually we are interested in the BPS states of the 4d black hole that is obtained by further compactifying the string along an S^1 . These states carry an additional quantum number q_0 corresponding to the momentum along the string. From the four dimensional perspective, the quantum numbers (p^a, q_a, q_0) are the numbers of D4, D2 and D0 branes in the type IIA compactification on X . In this paper we will be switching back and forth between a spacetime perspective from eleven, five, or even three dimensions, and a world volume perspective of the M5-brane or its reduction to a world sheet.

Before going to the world volume description of the M5-brane and its string reduction, let us describe the spectral flow of the BPS states from a spacetime perspective. The low

energy action of M-theory contains the Chern-Simons coupling²

$$S_{CS} = \int C \wedge F \wedge F, \tag{2.2}$$

where $F = dC = F^a \wedge \alpha_a$ is the four-form field strength. As a result, the M2-brane charge is defined as (here we work in 11D planck units)

$$q_a = \int_{S^2 \times S^1 \times D_a} (*F + C \wedge F). \tag{2.3}$$

The charge thus contains a Chern-Simons type contribution depending explicitly on the C -field. This term can be written as a volume integral of $F \wedge F$ and hence is invariant under small gauge transformations that vanish at infinity. However, it can still change under large gauge transformations corresponding to shifts in C by a closed and integral three form

$$C \rightarrow C - \sum_a k^a d\sigma \wedge \alpha_a \tag{2.4}$$

with $\sigma \in [0, 1]$ a coordinate for the S^1 . This transformation should be an exact symmetry of M-theory. The value of the charge q_a , though, is not invariant but instead receives an extra contribution proportional to the M5-brane charge p^a . Namely, using

$$\int_{D_a \times S^2 \times S^1} d\sigma \wedge \alpha_b \wedge F = d_{abc} \int_{S^2} F^c = d_{abc} p^c, \tag{2.5}$$

one finds that

$$q_a \rightarrow q_a - d_{abc} k^b p^c \tag{2.6}$$

under a large gauge transformation.³

It turns out that the q_0 quantum number also changes under the spectral flow. This is most easily seen from the world volume perspective that will be described in more details in the following section. In fact, one has [4]

$$q_0 \rightarrow q_0 + k^a q_a - \frac{1}{2} d_{abc} k^a k^b p^c. \tag{2.7}$$

It will turn out to be convenient to introduce the symmetric bilinear form

$$d_{ab} = -d_{abc} p^c = - \int_{\mathcal{P}} \alpha_a \wedge \alpha_b, \tag{2.8}$$

where α_a should be understood as the pullback of the harmonic 2-forms from the ambient Calabi-Yau X to the 4-cycle \mathcal{P} .

By the Lefschetz hyperplane theorem this form is non-degenerate. In fact it has signature $(b_2 - 1, 1)$ where $b_2 = \dim H^2(X)$ is the second Betti number of the Calabi-Yau space. Thus, for every positive divisor \mathcal{P} we obtain a natural metric d_{ab} on $H^2(X, \mathbb{Z})$ which turns

²Here and in the following we are somewhat sloppy in writing down the proper factors of 2π etc in effective actions. The final answer for the spectral flow is however correctly normalized.

³ d_{abc} is the cubic intersection number of the Calabi-Yau $d_{abc} = \int_X \alpha_a \wedge \alpha_b \wedge \alpha_c$.

it in to a Lorentzian lattice Λ . From (2.6) we see that the spectral flow transformation amounts to shifting q_a by an element of Λ .

The Dirac quantization condition suggests that the lattice of M2-brane charges coincides with $H^4(X, \mathbb{Z})$, which can be identified with the dual lattice Λ^* whose bilinear form d^{ab} is given by the inverse of d_{ab} . However, due to the Freed-Witten anomaly the M2-brane charge does not satisfy the usual Dirac quantization condition, but rather [12, 13]

$$q_a \in \frac{1}{2} d_{abc} p^b p^c \oplus \Lambda^*. \tag{2.9}$$

In terms of the bilinear form d_{ab} the flow equations (2.6) and (2.7) read

$$q_a \rightarrow q_a + d_{ab} k^b, \tag{2.10}$$

$$q_0 \rightarrow q_0 + k^a q_a + \frac{1}{2} d_{ab} k^a k^b. \tag{2.11}$$

In this form one can see explicitly that the collection of the shifts of M2 charges induced by all possible large gauge transformations is the Lorentzian lattice Λ and that the combination

$$\hat{q}_0 = q_0 - \frac{1}{2} d^{ab} q_a q_b \tag{2.12}$$

is invariant under the combined spectral flow of q_a and q_0 . Note that due to the integrality of the symmetric form d_{ab} one has $\Lambda \subset \Lambda^*$. In general, Λ is a proper subset of Λ^* , which means that not all charge configurations (Λ^*) are related to each other by spectral flow (Λ).

From the above argument we conclude that the combined spectral flow transformations (2.10) and (2.11) constitute a symmetry of M-theory/string theory. This gives a very non-trivial prediction on the BPS degeneracies that the number of BPS states $d_{p^a}(q_a, q_0)$ should be invariant under these transformations.

We can now compare this microscopic prediction with the macroscopic result. The leading macroscopic entropy of the 4d black hole with charges p^a , q_a and q_0 is given by [4]

$$S = 2\pi \sqrt{\hat{q}_0 D} \tag{2.13}$$

where

$$D \equiv \frac{1}{6} \int_X P \wedge P \wedge P = \frac{1}{6} d_{abc} p^a p^b p^c. \tag{2.14}$$

Note that $6D = -d_{ab} p^a p^b$ can be interpreted as the norm of the vector p^a . The entropy formula is consistent with our prediction that the entropy must be invariant under the spectral flow.

Finally, we would like to point out that the spectral flow (2.10) can be induced spontaneously by the nucleation of a M5/anti-M5 brane pair with magnetic charge k^a , where the M5 loops through the original (circular) M5 brane before annihilating again with the anti-M5 brane. We will make use of this comment in the next section where this same process is translated to the near horizon geometry.

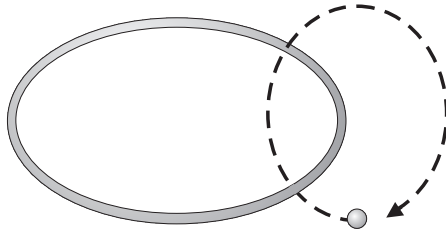


Figure 1: An M5 brane loops through the original (circular) M5 brane and then annihilates again with an anti-M5 brane.

2.2 The near-horizon geometry and reduction to three dimensions

In the decoupling near-horizon limit, the spacetime physics can be entirely captured by the world-volume theory of the brane. In this limit the 11-dimensional geometry becomes

$$X \times AdS_3 \times S^2, \quad (2.15)$$

with the Kähler moduli $k = t^a \alpha_a \in H^{1,1}(X)$ of the Calabi-Yau fixed by the attractor mechanism to be proportional to the charge vector p^a .

More explicitly, the attractor equation reads

$$\frac{t^a}{V^{1/3}} = \frac{p^a}{D^{1/3}},$$

where V denotes the volume of the Calabi-Yau

$$V = \frac{1}{6} \int_X k \wedge k \wedge k = \frac{1}{6} d_{abc} t^a t^b t^c. \quad (2.16)$$

Note that the volume V is not fixed by the attractor equation. Instead, the ratio V/D turns out to be related to the curvature radius ℓ of the AdS_3 and S^2 :

$$\frac{V}{D} \sim \frac{\ell_p^9}{\ell^3}.$$

The semi-classical limit of the M-theory corresponds to taking $\frac{\ell}{\ell_p} \rightarrow \infty$. To keep V/ℓ_p^6 large we need to take D very large as well. If we keep D fixed, then the CY becomes small and the 11-dimensional M-theory naturally reduces to a five-dimensional theory. For our purpose it will be useful to consider a further reduction along the compact S^2 to a three dimensional theory on the non-compact AdS_3 spacetime. In the low energy limit, this theory contains the metric and the $U(1)$ gauge fields A^a as the massless bosonic fields.

From the five-dimensional perspective, the 5-brane flux of M-theory background gets translated into a magnetic flux $F^a = dA^a$ of the $U(1)$ gauge fields through the S^2 :

$$\int_{S^2} F^a = p^a.$$

The eleven-dimensional Chern-Simons term of the C -field can therefore be reduced in two steps. First to five dimensions, where it takes the form

$$S_{CS} = \int_{AdS^3 \times S^2} d_{abc} A^a \wedge F^b \wedge F^c,$$

and subsequently, by integrating over the S^2 , to three dimensions, where it turns into the usual (Abelian) Chern-Simons action for the gauge fields on AdS_3 . In combination with the standard kinetic terms, we get

$$S = \frac{1}{16\pi G_3} \int \sqrt{g} (R - 2\ell^{-2}) + \frac{1}{g^2} \int g_{ab} F^a \wedge *F^b + \int d_{ab} A^a \wedge dA^b$$

as the terms in the bosonic action relevant for our discussion, where again the relative normalizations should be taken with a grain of salt and $g_{ab} = \int_X \alpha_a \wedge * \alpha_b$. The 3d Newton constant and the gauge coupling are given by

$$\frac{1}{G_3} \sim \frac{V\ell^2}{\ell_p^9} \sim \frac{D}{\ell}, \quad \frac{1}{g^2} \sim \frac{V\ell^2}{\ell_p^7} \sim \frac{D\ell_p^2}{\ell}. \quad (2.17)$$

We will end this section by some discussions about the spectral flow in the setting of the attractor geometry. First we note that the spectral flow argument in the previous section can be carried to the three-dimensional setting by dimensional reduction. The M2-brane charge q^a is defined now as an integral over a circle at spatial infinity of the AdS_3 as

$$q_a = \int_{S^1} \left(\frac{1}{g^2} *F_a + d_{ab} A^b \right).$$

Again one easily verifies that it changes as in (2.10) as a result of a large gauge transformation $A^a \rightarrow A^a + k^a d\sigma$ in three dimensions. The charge q_0 is related to the angular momentum in AdS_3 . To understand the shift in q_0 under spectral flow, one has to determine the contribution to the three-dimensional stress energy tensor due to the gauge field.

As mentioned above, the spectral flow has a nice physical interpretation in terms of the nucleation of an M5/anti-M5 brane pair. Let us now describe this process in the near horizon geometry. The following argument is most easily visualized by suppressing the (Euclidean) time direction and focusing on a spatial section of AdS_3 , which can be thought of as a copy of Euclidean AdS_2 and hence is topologically a disk. Together with the S^2 it forms a four dimensional space. First, recall that a wrapped M5 brane appears as a string-like object in this four dimensional space. Since an M5-brane is magnetically charged under the five-dimensional gauge fields A^a , it creates a "Dirac surface" of A^a . Of course, the location of the Dirac surface is unphysical and can be moved by a gauge transformation. Now suppose at a certain time an M5/anti-M5-brane pair nucleates in the center of AdS_2 in a way that the M5 and the anti-M5 branes both circle the equator of the S^2 . Subsequently, the M5 and the anti-M5 branes move in opposite directions on the S^2 , say the M5 brane to the north pole and anti-M5 to the south pole. In this way the M5 and anti-M5 brane pair creates a Dirac surface that stretches between them. Eventually both branes slip off and self-annihilate on the poles of the S^2 . What they leave behind now is a Dirac surface that wraps the whole S^2 and still sits at the origin of the AdS_2 . To remove it one literally has to move it from the center and take it to the spatial infinity. Once it crosses the boundary circle, its effect is to perform a large gauge transformation that is determined by the charge k^a of the M5 brane of the nucleated pair. We conclude that spectral flow can thus be induced by the nucleation of pairs of M5 and anti-M5 branes.

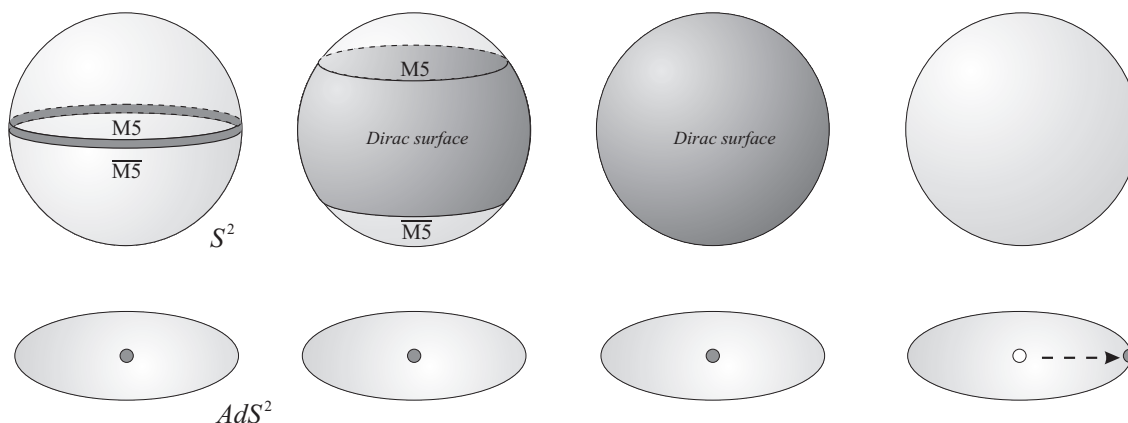


Figure 2: A large gauge transformation: (i) An M5-anti-M5 pair wrapping the equator of the S^2 nucleates at the center of the AdS_2 . (ii) The M5 and anti-M5 begin to move in the opposite directions in the S^2 , while still stay at the center of the AdS_2 . (iii) A Dirac surface wrapping the whole S^2 is formed. (iv) Finally one moves the Dirac surface from the bulk of the AdS_2 towards the spatial infinity across the boundary.

3. The (0,4) SCFT and its elliptic genus

The existence of the bound states of M2-branes to the M5-brane can be seen in an elegant way from the point of the view of the five-brane world-volume theory. This world-volume theory is a six-dimensional (0,2) superconformal field theory whose field content includes a tensor field with self-dual 3-form field strength H . The spacetime C field couples to H through the term

$$\int_Y C \wedge H \tag{3.1}$$

where $Y = \mathcal{P} \times S^1 \times \mathbb{R}$ denotes the world-volume of the five-brane.

In a bound state the M2-brane charges are dissolved into fluxes of H in the following way: the self-dual tensor H that carries the charges q_a has spatial components

$$H = d^{ab} q_a \alpha_b \wedge d\sigma, \tag{3.2}$$

with $\sigma \in [0, 1]$ being the coordinate along the S^1 . The timelike components follow from the self-duality condition. Combining the formulas (2.1) and (3.2), one sees that this produces the right coupling

$$\int_Y C \wedge H = q_a \int_{\mathbb{R}} A^a$$

of the U(1) gauge fields A^a to the charges q_a .

3.1 The (0,4) superconformal field theory

When we take the scale of the Calabi-Yau to be much smaller than the radius of the M-theory circle, the M5 world-volume theory naturally gets reduced along the 4-cycle \mathcal{P} to a two-dimensional conformal field theory with (0,4) supersymmetry. As usual the superconformal symmetries are identified with the supersymmetric isometries of the $AdS_3 \times$

S^2 manifold. This algebra contains the following generators that will be important for us: the right-moving stress-energy tensor \bar{T} , supercurrents \bar{G}_α^\pm and SU(2) R-symmetry currents $\bar{J}_{\alpha\beta}$. In particular the SU(2) R-symmetry corresponds to the rotations of the S^2 factor.

The self-dual tensor field H gives rise to a collection of chiral scalar fields φ^a through an expansion of H in harmonic forms. To be more specific, if we write its spatial components as

$$H = i^* \alpha_a \wedge \partial_\sigma \varphi^a d\sigma, \tag{3.3}$$

the self-duality condition⁴ automatically determines the temporal components. Here $i^* \alpha_a \in H^2(\mathcal{P}, \mathbb{Z})$ is the pull-back of the harmonic two-form α_a under the inclusion map $i : \mathcal{P} \hookrightarrow X$. The two-forms in $H^2(\mathcal{P}, \mathbb{Z})$ that are not in the image of i^* are not conserved due to M2-brane instantons and do not give rise to chiral scalar fields [4, 14]. Altogether one concludes that the self-dual and anti-self-dual two-forms on \mathcal{P} that do extend to non-trivial two-forms on X give rise to the left- and right-moving scalars in the CFT respectively. Using the fact that the Kähler form k is the only self-dual two form among all the harmonic 2-forms $i^* \alpha_a \in H^2(\mathcal{P}; \mathbb{Z})$, one finds one right-moving and $(b_2 - 1)$ left-moving bosons from the reduction of the self-dual tensor field H of the five-brane world volume theory.

In addition there are three non-chiral scalar fields x^i that describe the motion of the five-brane transverse to the Calabi-Yau. The four right-moving scalars assemble themselves into a $(0, 4)$ supermultiplet together with the four right-moving Goldstinos $\bar{\psi}_\alpha^\pm$ arising from the broken supersymmetry. This supermultiplet of free fields, which we will refer to as the "universal" supermultiplet [14], represents only a small part of the full conformal field theory, since it only gives a fixed (\mathcal{P} independent) contribution to the central charge. The bulk of the degrees of freedom are contained in a separate sector which can be interpreted as describing the deformation of the five-brane inside the Calabi-Yau. In the semi-classical limit this CFT will take the form of a heterotic sigma-model on the relevant moduli space. Especially, the left-moving and right-moving central charges are [4]

$$c_L = 6D + c_2 \cdot P, \quad c_R = 6D + \frac{1}{2} c_2 \cdot P. \tag{3.4}$$

For the following discussion it will be useful to spell out in more details the definition of the left-moving and right-moving components of the bosons φ^a . The fact that the Kähler form takes the form $k \sim [P] = p^a \alpha_a$ implies that there is only a single right-moving component corresponding to the component of φ^a in the direction of p^a . Let us therefore define the left- and right-moving projectors

$$P_{R,b}{}^a = -\frac{d_{bc} p^a p^c}{6D} = \delta_b^a - P_{L,b}{}^a. \tag{3.5}$$

One thus finds that the left-movers obey

$$P_{R,a}{}^b (\partial \varphi_L^a) = 0 \Leftrightarrow \partial \varphi_L^a = P_{L,b}{}^a (\partial \varphi^b) \tag{3.6}$$

⁴In the presence of a non-trivial background field C the self-duality equations are modified. This shows up in the CFT as Narain moduli for the chiral bosons. The dependence of the elliptic genus on the Narain moduli is easy to work out, they only appear in the theta functions (3.33) and in particular the entropy does not depend on them.

while the right-mover is defined through

$$\bar{\partial}\varphi_R^a = P_{R,b}{}^a(\bar{\partial}\varphi^b). \tag{3.7}$$

The charges q_a correspond to the momenta of the scalars φ^a , thus a membrane of charge q_a is represented in the CFT by an insertion of the vertex operator

$$V_q = \exp i(q_{L,a}\varphi_L^a + q_{R,a}\varphi_R^a), \tag{3.8}$$

where q_L and q_R are defined in terms of the (half) integral charges q_a as

$$q_{L,a} = P_{L,a}{}^b q_b, \quad q_{R,a} = P_{R,a}{}^b q_b. \tag{3.9}$$

Furthermore one has [4]

$$q_L^2 - q_R^2 = d^{ab} q_a q_b. \tag{3.10}$$

Later we will be interested in the supersymmetric ground states in the charge sectors labelled by q_a . A special role in this analysis will be played by the right-moving universal multiplet. It will be convenient to introduce the right-moving field by $\bar{\varphi}$ defined by

$$\varphi_R^a = p^a \bar{\varphi}. \tag{3.11}$$

It is normalized so that its operator product expansion reads

$$\bar{\partial}\bar{\varphi}(\bar{z}) \cdot \bar{\partial}\bar{\varphi}(\bar{w}) \sim -\frac{1}{6D} \frac{1}{(\bar{z} - \bar{w})^2}$$

We also normalize the Goldstinos $\bar{\psi}_\alpha^\pm$ which sit in the same supermultiplet in a similar fashion, so that the following identities holds

$$\begin{aligned} \epsilon^{\alpha\beta} \left\{ \bar{\psi}_{\alpha,0}^+, \bar{\psi}_{\beta,0}^- \right\} &= \frac{1}{3D}, \\ \epsilon^{\alpha\beta} \left\{ \bar{G}_{\alpha,0}^+, \bar{\psi}_{\beta,0}^- \right\} &= \frac{p^a q_a}{3D} \\ \epsilon^{\alpha\beta} \left\{ \bar{G}_{\alpha,0}^+, \bar{G}_{\beta,0}^- \right\} &= 4 \left(\bar{L}_0 - \frac{c_R}{24} \right). \end{aligned} \tag{3.12}$$

The spectral flow relations (2.10) (2.11) are implemented in the (0,4) CFT as a symmetry of the superconformal algebra. It is given by

$$\begin{aligned} L_m &\rightarrow L_m + k_L^a Q_{aL,m} - \frac{1}{2} d_{abc} k_L^a k_L^b p^c \delta_{m,0} \\ Q_{aL,m} &\rightarrow Q_{aL,m} - d_{abc} k_L^b p^c \delta_{m,0}, \end{aligned} \tag{3.13}$$

where $Q_{aL,m}$ are the modes of the $(b_2 - 1)$ left-moving bosons, and with similar expressions for \bar{L}_m and $Q_{aR,m}$. Note that these transformations leave $L_0 - \frac{1}{2}Q_L^2$ and $\bar{L}_0 - \frac{1}{2}Q_R^2$ invariant.

3.2 A generalized elliptic genus

The generating function of BPS bound states for a fixed M5-brane charge p^a can be identified with a generalized elliptic genus of the CFT (see also [9–11]). More precisely, we want to compute the partition function

$$Z'_{p^a}(\tau, \bar{\tau}; y^a) = \text{Tr} \left[F^2 (-1)^F e^{\pi i p^a q_a} e^{2\pi i \tau (L_0 - \frac{c_L}{24})} e^{-2\pi i \bar{\tau} (\bar{L}_0 - \frac{c_R}{24})} e^{2\pi i y^a q_a} \right], \quad (3.14)$$

where F denotes the fermion number $F = 2\bar{J}_0^3$, with \bar{J}_0^3 one of the generators of $SU(2)_R$. The appearance of the extra phase $e^{\pi i p^a q_a}$ is a bit subtle and will be explained below. Notice that we could also have chosen to absorb this phase in the definition of fermion number, but instead we will keep it explicit in what follows. The reason that we have to insert F^2 inside the trace is to absorb the four fermion zero-modes and therefore obtain a non-zero expression [16]. The trace with an insertion of F^k with $k = 0, 1$ vanishes identically.

Only short multiplets, namely those states that are annihilated by four combinations of world-sheet supercharges $\bar{G}_{\alpha,0}^\pm$ shifted by an appropriate multiple of the fermion zero-mode $\bar{\psi}_{\alpha,0}^\pm$, contribute to the partition function (3.14). In the sector with $U(1)$ charges q_a , these states satisfy

$$\left(\bar{G}_{\alpha,0}^\pm - p^a q_a \bar{\psi}_{\alpha,0}^\pm \right) |BPS\rangle = 0. \quad (3.15)$$

Using the fact that the norm of these states is positive, one derives that

$$\epsilon^{\alpha\beta} \left\{ \left(\bar{G}_{\alpha,0}^+ - p^a q_a \bar{\psi}_{\alpha,0}^+ \right), \left(\bar{G}_{\beta,0}^- - p^a q_a \bar{\psi}_{\beta,0}^- \right) \right\} |BPS\rangle = 4 \left(\bar{L}_0 - \frac{1}{2} q_R^2 - \frac{c_R}{24} \right) |BPS\rangle = 0. \quad (3.16)$$

where

$$q_R^2 \equiv -d^{ab} q_{R,a} q_{R,b} = \frac{(p^a q_a)^2}{6D}. \quad (3.17)$$

It is straightforward to show that once we include the three-momentum \vec{p} of the non-compact scalars of the universal supermultiplet, the BPS condition becomes

$$\bar{L}_0 = \frac{1}{2} q_R^2 + \frac{1}{2} \vec{p}^2 + \frac{c_R}{24}.$$

In each BPS multiplet, there are four linear combinations of $\bar{G}_{\alpha,0}^\pm$ and $\bar{\psi}_{\alpha,0}^\pm$ that act non-trivially and produce a short multiplet of 4 BPS states. A direct consequence of this is that the partition function depends in a specific way on $\bar{\tau}$. In the sector with $\vec{p} = 0$ this dependency is captured by the “heat equation”

$$\left[\partial_{\bar{\tau}} + \frac{1}{24\pi i D} (p^a \partial_{y^a})^2 \right] Z'_{p^a}(\tau, \bar{\tau}, y) = 0, \quad (3.18)$$

which means that the dependence on $\bar{\tau}$ is completely under control. It turns out that it only appears in the elliptic genus through certain theta functions, which is discussed in detail in appendix A.

Eventually our aim is to show that the partition function has an asymptotic expansion in terms of semi-classical saddle-points of the three-dimensional supergravity theory. A crucial property of $Z'_{p^a}(\tau, \bar{\tau}; y^a)$ for this is that it is a modular form. To be precise, it is

a modular form of weight $(0, 2)$. To show this, let us introduce a generalized partition function

$$W_{p^a}(\tau, \bar{\tau}; y^a, z) = \text{Tr} \left[e^{2\pi i z F} e^{\pi i p \cdot q} e^{2\pi i \tau (L_0 - \frac{c_L}{24})} e^{-2\pi i \bar{\tau} (\bar{L}_0 - \frac{c_R}{24})} e^{2\pi i y^a q_a} \right] \quad (3.19)$$

so that

$$Z'_{p^a}(\tau, \bar{\tau}; y^a) = -\frac{1}{4\pi^2} \partial_z^2 W_{p^a}(\tau, \bar{\tau}; y^a, z)|_{z=1/2}.$$

The partition function $W_{p^a}(\tau, \bar{\tau}; y^a, z)$ can be represented as some kind of “functional integral” over all degrees of freedom of the (0,4) CFT on a torus with modular parameter τ and certain Wilson lines parametrized by y^a and z . As such it should be independent of the choice of cycles on the torus, and hence be invariant under the modular transformations

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \bar{\tau} \rightarrow \frac{a\bar{\tau} + b}{c\bar{\tau} + d}, \quad y_L^a \rightarrow \frac{y_L^a}{c\tau + d}, \quad y_R^a \rightarrow \frac{y_R^a}{c\bar{\tau} + d}, \quad z \rightarrow \frac{z}{c\bar{\tau} + d},$$

where y_L, y_R are the projected potentials $y_L^a = P_{L,b}{}^a y^b$, $y_R^a = P_{R,b}{}^a y^b$. This proves that $Z'_{p^a}(\tau, \bar{\tau}; y^a)$ has weight $(0, 2)$.

As mentioned above, the partition function $Z'_{p^a}(\tau, \bar{\tau}; y^a, z)$ contains a continuous degeneracy in the BPS states due to the zero modes in the \mathbb{R}^3 part of the $S^1 \times \mathbb{R}^3$, $\mathcal{N} = (0, 4)$ “universal” multiplet. We wish to extract this degeneracy, and this can be done by defining

$$Z'_{p^a}(\tau, \bar{\tau}, y^a, z) = \int d^3 p (e^{2\pi i \tau} e^{-2\pi i \bar{\tau}})^{\frac{1}{2} \vec{p}^2} Z_{p^a}(\tau, \bar{\tau}, y^a, z) \quad (3.20)$$

where Z_{p^a} is the index without zero modes.⁵ The integral is proportional to $\text{Im}(\tau)^{-3/2}$ which has weight $(\frac{3}{2}, \frac{3}{2})$. So we establish that Z_{p^a} has weight $(-\frac{3}{2}, \frac{1}{2})$.

From (3.18), or equivalently, (3.16), we see that we can always reconstruct the $\bar{\tau}$ -dependence of the elliptic genus from the y dependence. Physically this corresponds to the fact that for BPS states, \bar{L}_0 is determined uniquely by the $U(1)$ charges q_a . Therefore, for clarity we will set $\tau = \bar{\tau}$ in the following part of the paper without losing any generality, as the full computation with $\bar{\tau}$ -dependence incorporated will be given in the appendix.

For the purpose of extracting the degeneracy, we would like to write Z_{p^a} in a Fourier expansion as

$$Z_{p^a}(\tau, y) = \sum_{q_0; q_a \in \Lambda^* + d_{ab} p^b / 2} d(q_0, q) e^{2\pi i \tau q_0} e^{2\pi i y^a q_a}. \quad (3.21)$$

In more detail, for supersymmetric states one has⁶

$$\begin{aligned} L_0 &= q_0^{\text{ind}} + \frac{d^{ab} q_{L,a} q_{L,b}}{2} \\ \bar{L}_0 - \frac{c_R}{24} &= -\frac{d^{ab} q_{R,a} q_{R,b}}{2} \\ \Rightarrow q_0 &= \left(L_0 - \frac{c_L}{24} \right) - \left(\bar{L}_0 - \frac{c_R}{24} \right) = q_0^{\text{ind}} + \frac{d^{ab} q_a q_b}{2} - \frac{c_L}{24}, \end{aligned} \quad (3.22)$$

⁵Here the p 's refer to the momenta in \mathbb{R}^3 , not to be confused with the M5 brane charges p^a .

⁶The appearance of the extra $\frac{c_R}{24}$ is due to the fact that we are computing the elliptic genus in the R sector of the CFT, while this term will be absent on the right-moving supersymmetric side in the NS sector, where the supergravity computation is carried out.

where q_0^{ind} is the induced D0-brane charge. This need not be an integer but supersymmetry requires $q_0^{\text{ind}} \geq 0$. As argued in section 3.1, large gauge transformations shift the charges as (2.10), (2.11). This should be a symmetry of M-theory and leave the degeneracies of the BPS states invariant. In the language of the dual CFT, these large gauge transformations correspond to spectral flows which flow the $U(1)$ charges as (2.10) while keeping

$$\hat{q}_0 \equiv q_0 - \frac{1}{2}d^{ab}q_aq_b = q_0^{\text{ind}} - \frac{c_L}{24} \tag{3.23}$$

invariant. This is a consequence of the fact that the spectral flow operation, while shifting the zero-modes of the $U(1)$ currents and thus shifting their contribution to the momentum along the S^1 , permutes with all the other operators in the CFT and thus leaves the rest of the S^1 momentum unchanged.

An important subtlety is that the appearance of the extra phase $e^{\pi ip \cdot q}$ in (3.14), on top of $(-1)^F$. This additional phase implies that under spectral flow with charge vector k^a , states in the Ramond sector pick up an extra phase $(-1)^{p \cdot k}$. This is related to the shift $p_a/2 = d_{ab}p^b/2$ in the charges q_a due to the Freed-Witten anomaly. This can be seen, for example, from the operator algebra. Consider the vertex operators $V_{k^a}(z)$ acting on the state $|q\rangle$ with $q_a \in \Lambda^* + p_a/2$, one has

$$V_{k^a}(z) |q\rangle = z^{k \cdot q} |q + k\rangle . \tag{3.24}$$

The OPE will pick up a phase $\exp(2\pi ik \cdot q) = (-1)^{k \cdot p}$ when z circles around the origin. Locality of the OPE requires projection onto the states with even $k \cdot p$, which explains why the elliptic genus needs to contain a factor $(-1)^{p \cdot k}$ for it to be modular invariant. For convenience we include however a factor $e^{\pi ip \cdot q} = e^{\pi ip \cdot \mu} e^{\pi ip \cdot k}$ in our definition of the elliptic genus. The term $p \cdot \mu$ could be interpreted as an additional overall phase or as a fractional contribution to the fermion number.

Substitution of the transformations (2.10) and (2.11) in (3.21) shows directly the transformation property of the partition function under spectral flow to be

$$Z_{p^a}(\tau, y) = e^{\pi ip \cdot k} e^{\pi i \tau k^2 + 2\pi i y \cdot k} Z_{p^a}(\tau, y + k\tau), \tag{3.25}$$

where we have taken the fermion number as well into account. Spectral flow therefore predicts⁷

$$d(q_0, q) = e^{\pi ip \cdot k} d\left(q_0 + k \cdot q + \frac{k^2}{2}, q + k\right) . \tag{3.26}$$

From this invariance property and the shift of the charge lattice due to the anomaly (2.9), it turns out to be convenient to decompose the $U(1)$ charges as

$$q = \mu + k \quad ; \quad \mu \in \Lambda^*/\Lambda + \frac{p}{2} \quad ; \quad k \in \Lambda . \tag{3.27}$$

In this way we split the sum over the charges into a sum over μ in the "fundamental domain" $\Lambda^*/\Lambda + \frac{p}{2}$ and a sum over the different possible spectral flows Λ . Exploiting (3.26), we now

⁷To keep the transparency of the equations, we leave out the indices in various places in this section. But it should be understood that all the indices are lowered, raised, and contracted by using d_{ab} and d^{ab} .

rewrite the elliptic genus as

$$Z_{p^a}(\tau, y) = \sum_{\mu \in \Lambda^*/\Lambda + \frac{p}{2}} \sum_{k \in \Lambda} \sum_{-\frac{c_L}{24} \leq \hat{q}_0} e^{\pi i p \cdot \mu} e^{\pi i p \cdot k} d_\mu(\hat{q}_0) e^{2\pi i \tau \left(\hat{q}_0 + \frac{(\mu+k)^2}{2} \right)} e^{2\pi i y \cdot (\mu+k)}, \quad (3.28)$$

where $d_\mu(\hat{q}_0) = (-1)^{k \cdot p} d\left(\hat{q}_0 + \frac{(\mu+k)^2}{2}, \mu + k\right)$. Putting all the factors together, we conclude that our generalized elliptic genus can be decomposed into purely holomorphic modular forms which contain all the information about the degeneracies and theta-functions (this is in fact also a general property of weak Jacobi forms, see e.g. [15])

$$Z_{p^a}(\tau, y) = \sum_{\mu \in \Lambda^*/\Lambda + \frac{p}{2}} \chi_\mu(\tau) \theta_\mu(\tau, y), \quad (3.29)$$

where

$$\chi_\mu(\tau) = \sum_{-\frac{c_L}{24} \leq \hat{q}_0} d_\mu(\hat{q}_0) e^{2\pi i \tau \hat{q}_0} \quad (3.30)$$

$$\theta_\mu(\tau, y) = \sum_{k \in \Lambda} e^{\pi i p \cdot \mu} e^{\pi i p \cdot k} e^{2\pi i \tau \frac{(\mu+k)^2}{2} + 2\pi i y \cdot (\mu+k)}. \quad (3.31)$$

Especially, the degeneracies in a sector with given q_a are contained purely in $\chi_\mu(\tau)$, as there is only one term in the theta functions that contributes for each value of q_a . And the necessity of the factor $(-1)^{p \cdot (\mu+k)}$ can also be seen from the fact that it ensures that the theta functions transform into themselves under modular transformations.

Since the lattice Λ is Lorentzian, the holomorphic theta functions are only defined formally, since the sum is not convergent. To obtain a convergent expression one has to restore the $\bar{\tau}$ -dependence. We will discuss this in more detail in the appendix, but for completeness we give here the resulting $\bar{\tau}$ -dependent partition function. It takes the form

$$Z_{p^a}(\tau, \bar{\tau}, y_L, y_R) = \sum_{\mu \in \Lambda^*/\Lambda + \frac{p}{2}} \chi_\mu(\tau) \theta_\mu(\tau, \bar{\tau}, y_L, y_R), \quad (3.32)$$

where χ_μ is defined as above. Hence, the only difference is that the theta-function is replaced by a so-called Siegel-Narain theta function:

$$\theta_\mu(\tau, \bar{\tau}, y_L, y_R) = \sum_{k \in \Lambda} e^{\pi i p \cdot \mu} e^{\pi i p \cdot k} e^{2\pi i \tau \frac{(\mu_L+k_L)^2}{2}} e^{-2\pi i \bar{\tau} \frac{(\mu_R+k_R)^2}{2}} e^{2\pi i y \cdot (\mu+k)}, \quad (3.33)$$

which is convergent due to the presence of $\bar{\tau}$ -dependence. A discussion of the modular properties of these theta functions and a further explanation of these expressions can be found in the appendix and in e.g. [23].

4. The Farey tail expansion of the elliptic genus

Now that we have obtained an expression for the elliptic genus, as the following step we would like to rewrite it in the form of an asymptotic expansion that is suitable for a

semi-classical interpretation. The main tool we will use for this purpose is the so-called Rademacher formula. This formula can be applied to practically any modular form and is rather insensitive to most of the details of the system. Therefore, to exhibit the main idea we will in this first subsection present the general arguments that lead to the Rademacher formula. Subsequently we will generalize it so that it can be applied to the case of interest.

4.1 The Rademacher formula

An essential ingredient of the Farey tail is the Rademacher formula, which is an asymptotic expansion given by a sum over images of the "polar part" of the partition function under the modular group, which we will define shortly. The name "Farey tail" comes from the fact that the terms in the expansion corresponding to different images under the modular group are labelled by a Farey series, which is a sequence of rational numbers ordered by the size of the denominator. One term is the leading term of the sequence whereas the others form a "tail" of subleading corrections.

To illustrate the basic idea of the Rademacher formula, let us consider, as the first case, a modular form $Z(\tau)$ of weight w with Laurent expansion

$$Z(\tau) = \sum_{n \geq 0} d(n) e^{2\pi i(n - \frac{c}{24})\tau}. \tag{4.1}$$

Here $d(n)$ are integer degeneracies. The fact that it has weight w means that under modular transformations it transforms as

$$Z(\tau) = Z\left(\frac{a\tau + b}{c\tau + d}\right) (c\tau + d)^{-w}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}). \tag{4.2}$$

The modular form $Z(\tau)$, when expressed in terms of $q = \exp 2\pi i\tau$, has poles at the 'origin' $q = 0$ and its images under the modular group.

The Rademacher formula is obtained by first truncating the sum over n to include only the polar terms, that is, those terms with $n < \frac{c}{24}$, and subsequently summing over all the images under the modular group. This procedure reproduces the original modular form $Z(\tau)$, since by construction one obtains a modular form with the same weight and poles at exactly the same location. The Rademacher formula thus reads

$$Z(\tau) = \sum_{\gamma \in \Gamma/\Gamma_\infty} Z^-\left(\frac{a\tau + b}{c\tau + d}\right) (c\tau + d)^{-w} \tag{4.3}$$

with

$$Z^-(\tau) = \sum_{0 \leq n < \frac{c}{24}} d(n) e^{2\pi i\tau(n - \frac{c}{24})}. \tag{4.4}$$

Here $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and Γ_∞ denotes the modular subgroup that keeps $\tau = i\infty$ fixed and is generated by $\tau \rightarrow \tau + 1$. To ensure that the sum over the modular group is convergent, the modular weight w of $Z(\tau)$ should be positive.

The Rademacher formula can be used to express the degeneracies $d(n)$ of arbitrary n in terms of those corresponding to the polar part, and this leads to an exact version of the Cardy formula. In integral form it reads

$$d(m) = \sum_{0 \leq n < \frac{c}{24}} d(n) \sum_{\gamma \in \Gamma_\infty \setminus \Gamma' / \Gamma_\infty} K_\gamma(n; m) \tag{4.5}$$

with

$$K_\gamma(n; m) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau e^{2\pi i \left(\frac{a\tau+b}{c\tau+d} \left(n - \frac{c}{24} \right) - \tau \left(m - \frac{c}{24} \right) \right)} (c\tau + d)^{-w} , \tag{4.6}$$

where the prime of Γ' indicates that we leave out the identity element of the modular group. Here we have performed part of the summation over the modular group to extend the range of integration over τ to the full real axis. The resulting integral can be expressed in terms of generalized Bessel functions. For more details, see [25].

Both for the D1-D5 system as well as for the present case of attractor black holes, one needs a generalization of the Rademacher formula, namely one that applies to vector-valued modular forms (or characters) whose transformation properties involve a non-trivial representation of the modular group. In both of these cases one encounters characters $\chi_\mu(\tau)$ with an expansion of the form

$$\chi_\mu(\tau) = \sum_n d_\mu(n) e^{2\pi i \tau \left(n + \Delta_\mu - \frac{c}{24} \right)} \tag{4.7}$$

in terms of integral coefficients $d_\mu(n)$. The modular transformation rules take the form

$$\chi_\mu(\tau) = M(\gamma)_\mu^\nu \chi_\nu \left(\frac{a\tau + b}{c\tau + d} \right) (c\tau + d)^{-w_\chi} \tag{4.8}$$

where the matrices $M(\gamma)$ form a faithful representation of the modular group and w_χ is the weight of the character χ_μ . In this case the conformal weights Δ_μ are not necessarily integers anymore. By applying a similar reasoning as above, one obtains a generalized Rademacher formula involving the matrices $M(\gamma)$. In full glory it reads

$$\chi_\mu(\tau) = \sum_{\gamma \in \Gamma / \Gamma_\infty} M(\gamma)_\mu^\nu \sum_{n + \Delta_\nu < \frac{c}{24}} d_\nu(n) e^{2\pi i \frac{a\tau+b}{c\tau+d} \left(n + \Delta_\nu - \frac{c}{24} \right)} (c\tau + d)^{-w_\chi} \tag{4.9}$$

We eventually want to apply the formula to an elliptic genus $Z(\tau, y)$ that in addition to the modular parameter τ depends on one or more potentials (or wilson lines) y^A , $A = 1, \dots, r$. The crucial property of the elliptic genus that allows one to apply the Rademacher formula, is that it factorizes in terms of theta functions as

$$Z(\tau, y) = \sum_\mu \chi_\mu(\tau) \theta_\mu(\tau, y).$$

The rank of the theta function equals the number r of the potentials y^A . A necessary and sufficient condition for $Z(\tau, y)$ to factorize in this way is that it obeys the following "spectral flow" property

$$Z(\tau, y + n\tau + m) = e^{-i\pi n^2 \tau - 2\pi i n y} Z(\tau, y)$$

and it transforms under $\tau \rightarrow \frac{a\tau+b}{c\tau+d}$ and $y \rightarrow \frac{y}{c\tau+d}$ as

$$Z\left(\frac{a\tau+b}{c\tau+d}, \frac{y}{c\tau+d}\right) = e^{\frac{i\pi cy^2}{c\tau+d}} Z(\tau, y)(c\tau+d)^w$$

These transformation properties imply that $Z(\tau, y)$ is a weak Jacobi-form of weight w . Given the fact that the theta functions $\theta_\mu(\tau, y)$ form a representation of the modular group and have modular weight equal to $\frac{1}{2}r$, it follows that the characters $\chi_\mu(\tau)$ transform in the conjugate (=inverse) representation and have a modular weight

$$w_\chi = w - \frac{1}{2}r .$$

This fact can now be used to write the Rademacher formula for the full partition function as

$$Z(\tau, y) = \sum_{\gamma \in \Gamma/\Gamma_\infty} e^{-\pi i \frac{cy^2}{c\tau+d}} \chi_\mu^- \left(\frac{a\tau+b}{c\tau+d} \right) \theta_\mu \left(\frac{a\tau+b}{c\tau+d}, \frac{y}{c\tau+d} \right) (c\tau+d)^{-w} \quad (4.10)$$

The Rademacher expansion is primarily acting on the vector valued modular forms $\chi_\mu(\tau)$ when applied to a weak Jacobi form. In case $w_\chi < 0$, first a so called ‘‘Farey tail transform’’ [25] has to be done to get a convergent answer, given by

$$\tilde{\chi}_\mu(\tau) = \left(q \frac{\partial}{\partial q}\right)^{1-w_\chi} \chi_\mu(\tau) . \quad (4.11)$$

The transformed function has weight $\tilde{w}_\chi = 2 - w_\chi$, which shows that the Rademacher expansion can be applied to this modular form in case $w_\chi < 0$.

4.2 Application to the (0,4) elliptic genus

We now like to apply the Rademacher formula to the elliptic genus of the (0,4) SCFT and obtain an exact rewriting of the generating function $Z_{p^a}(\tau, y)$ of the BPS degeneracies $d_\mu(n)$. As stated above, one needs $\chi_\mu(\tau)$ to have a weight $w_\chi > 0$ for the expansion to converge [25]. $Z_{p^a}(\tau, y)$ has a negative weight $-\frac{3}{2} + \frac{1}{2} = -1$. From which we deduce that $\chi_\mu(\tau)$ has weight $-1 - \frac{1}{2}r$, which is manifestly negative. In order to get a meaningful and convergent Rademacher expansion, we apply the Farey tail transformation eq. (4.11). This gives a transformed modular form with weight $\tilde{w}_\chi = 3 + \frac{1}{2}r$. Combining it again with the theta functions gives a weak Jacobi form $\tilde{Z}_{p^a}(\tau, y)$ of weight $w = 3 + r$.

The full answer is of course similar to (4.10), and reads

$$\tilde{Z}_{p^a}(\tau, y) = \sum_{\mu \in \Lambda^*/\Lambda + \frac{p}{2}} \sum_{\gamma \in \Gamma/\Gamma_\infty} e^{-\pi i \frac{cy^2}{c\tau+d}} \tilde{\chi}_\mu^- \left(\frac{a\tau+b}{c\tau+d} \right) \theta_\mu \left(\frac{a\tau+b}{c\tau+d}, \frac{y}{c\tau+d} \right) (c\tau+d)^{-(3+r)} , \quad (4.12)$$

where

$$\tilde{\chi}_\mu^-(\tau) = \sum_{-\frac{c\tau}{24} \leq \hat{q}_0 < 0} \tilde{d}_\mu(\hat{q}_0) e^{2\pi i \tau \hat{q}_0}, \quad \tilde{d}_\mu(\hat{q}_0) = (\hat{q}_0)^{2+\frac{1}{2}r} d_\mu(\hat{q}_0), \quad (4.13)$$

now contains only the "polar part". This is our Attractor Farey Tail: the Rademacher expansion for the elliptic genus which is the generating function of $\mathcal{N} = 2$ D4-D2-D0 BPS black hole degeneracies. Notice that both the Farey tail transform as well as the Rademacher expansion commute with putting $\tau = \bar{\tau}$. In other words, if we would have kept the $\bar{\tau}$ dependence all along and would only have put $\tau = \bar{\tau}$ at the end we still would have ended up with the same result (4.12).

5. Spacetime interpretation of the attractor Farey tail

So far the Rademacher formula appears to be just a mathematical result. What makes it interesting is that it has a very natural interpretation from the point of view of a dual gravitational theory. In this section we discuss the interpretation of the Farey tail expansion first in terms of the effective supergravity action, and subsequently from an M-theory/string theory perspective. We will first discuss the gravitational interpretation of the general formula presented in section 4.1, and then turn to the present case of the attractor black holes.

5.1 Gravitational interpretation of the Rademacher formula

Microscopic systems described by a 2d CFT have a dual description in terms of a string- or M-theory on a space that contains AdS_3 as the non-compact directions. This is because AdS_3 is the unique space whose isometry group is identical to the 2d conformal group. The miracle of AdS/CFT is that the dual theory contains gravity, which suggests that the partition function of the 2d CFT somehow must have an interpretation as a sum over geometries. The full dual theory is defined on a space that is 10- or 11-dimensional, but except for the three directions of AdS_3 these dimensions are all compact. Hence, by performing a dimensional reduction along the compact directions we find that the dual theory can be represented as a (super-)gravity theory on AdS_3 . The effective action therefore contains the Einstein action for the 3d metric

$$S_E = \frac{1}{16\pi G_3} \int_{AdS} \sqrt{g}(R - 2\ell^{-2}) + \frac{1}{8\pi G_3} \int_{\partial(AdS)} \sqrt{h}(K - \ell^{-1})$$

where we have included the Gibbons-Hawking boundary term. Here ℓ represents the AdS-radius. According to the AdS/CFT dictionary, the 3d Newton constant G_3 is related to the central charge c of the CFT by

$$\frac{3\ell}{2G_3} = c. \tag{5.1}$$

The dictionary also states that the partition function $Z(\tau)$ of the CFT is equal to that of the dual gravitational theory on (a quotient of) AdS_3 , whose boundary geometry coincides with the 2d torus on which the CFT is defined. The shape of the torus is kept fixed and parametrized by the modular parameter τ .

The rules of quantum gravity tell us to sum over all possible geometries with the same asymptotic boundary conditions. For the case at hand, this means that we have to sum over all possible three dimensional geometries with the torus as the asymptotic boundary.

Semi-classically, these geometries satisfy the equations of the motion of the supergravity theory, and hence are locally AdS_3 . There indeed exists an Euclidean three geometry with constant curvature which has T^2 with modular parameter τ as its boundary. It is the BTZ black hole, which is described by the Euclidean line element

$$ds^2 = N^2(r)dt_E^2 + \ell^{-2}N^{-2}(r)dr^2 + r^2(d\phi + N_\phi(r)dt_E)^2$$

with

$$N^2(r) = \frac{(r^2 - \tau_2^2)(r^2 + \tau_1^2)}{r^2}, \quad N_\phi(r) = \frac{\tau_1\tau_2}{r^2}.$$

Here $\tau = \tau_1 + i\tau_2$ is the modular parameter of the boundary torus. Using (5.1), one can compute the Euclidean action of this solution and obtain [17]

$$S = -\frac{\pi c}{6} \text{Im} \frac{1}{\tau}.$$

For the present purpose of counting BPS states, one needs to consider extremal BTZ black holes. With the Minkowski signature this means that its mass and angular momentum are equal. After analytic continuation to a Euclidean complexified geometry, one finds that the action has become complex and equals $i\pi \frac{c}{12}\tau$.

Note that a torus with modular parameter τ is equivalent to a torus with parameter $\frac{a\tau+b}{c\tau+d}$, since they differ only by a relabelling of the A - and B -cycles. But the Euclidean BTZ solution labelled by $\frac{a\tau+b}{c\tau+d}$ in general differs from the one labelled by τ , with the difference being that these three-dimensional geometries fill up the boundary torus in distinct ways. Namely, for the above BTZ solution the torus is filled in such a way that its A -cycle is contractible. After a modular transformation, this would become the $\gamma(A) = cA + dB$ cycle. In fact, the BTZ black hole is related to thermal AdS_3 with metric

$$ds^2 = (r^2 + \ell^2)dt_E^2 + \frac{dr^2}{r^2 + \ell^2} + r^2d\phi^2 \tag{5.2}$$

after interchanging the A - and B -cycles and with t_E and ϕ periodically identified as

$$t_E \equiv t_E + 2\pi n\tau_2, \quad \phi \equiv \phi + 2\pi n\tau_1.$$

In this case the B -cycle is non-contractible, while the A -cycle is now contractible. Notice that in this metric it is manifest that $\tau \rightarrow \tau + 1$ gives the same geometry. The Euclidean action for this geometry is $S = i\pi \frac{c}{12}\tau$.

The classical geometries with a given boundary torus with modular parameter τ can now be obtained from either the thermal AdS background or the BTZ background by modular transformations. For definiteness let us take the thermal AdS as our reference point, so that the classical action for the geometry obtained by acting with an element γ of the modular group is

$$S = i\pi \frac{c}{12} \left(\frac{a\tau + b}{c\tau + d} \right).$$

One easily recognizes that these solutions precisely give all the leading contributions in the Farey tail expansion corresponding to $n = 0$ terms in (4.1). In fact, these terms occur

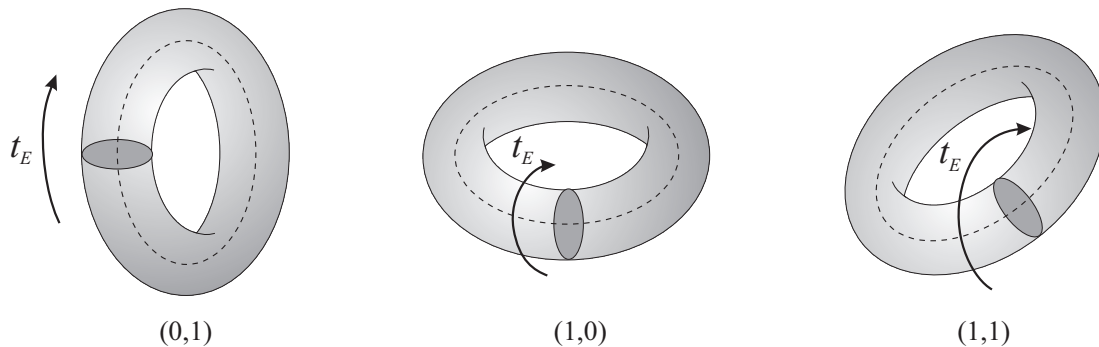


Figure 3: (i) Thermal AdS_3 , with the B-cycle being contractible. (ii) BTZ black hole, with the A-cycle being contractible. (iii) The geometry with the (A+B)-cycle being contractible.

with a multiplicity one since they represent the vacuum of the SCFT. The other terms in the expansion should then be regarded as dressing the Euclidean background with certain contributions that change the energy of the vacuum. Already in the original Farey tail paper these contributions were interpreted as coming from certain (virtual) particles that circle around the non-contractible cycle. In fact, in the next section we will give a further justification of this interpretation for the case of the attractor black holes. Specifically, by using the arguments of Gaiotto et al. [8], we find that the subleading contributions are due to a gas of wrapped M2-branes which carry quantum numbers corresponding to the charges and the spin in the AdS_3 geometry. The truncation to the polar terms can in turn be interpreted as imposing the restriction that the gas of particles are not heavy enough yet to form a black hole. The AdS_3 geometry carries a certain negative energy which allows a certain amount of particles to be present without causing gravitational collapse. However, when the energy surpasses a certain bound then a black hole will form through a Hawking-Page transition. In the case of the AdS_3/CFT_2 correspondence such an interpretation was first proposed by Martinec [18]. Indeed, when one compares the elliptic genus computed from supergravity to the exact CFT answer, one finds a mismatch at the threshold for black hole formation [19], see also [25].

The only feature of the Rademacher formula that really depends on the system at hand is the theta function. The origin of the theta function lies in the chiral bosons on the world sheet of the string reduction of the M5-brane. From a spacetime perspective, however, these theta functions are due to the gauge fields and in particular have their origin in the Chern-Simons term as discussed in section 2. In fact, the partition function of a spacetime effective theory that includes precisely such Chern-Simons terms in addition to the usual Yang-Mills action was analyzed in detail by Gukov et al. in [20]. These authors showed that the partition function indeed decomposes into a sum of Siegel-Narain theta functions. The $\bar{\tau}$ -dependence arises because one of the gauge field components is treated differently from the others, to ensure that the partition function indeed converges. Formally it is possible to treat all gauge fields in the same way, and choose boundary conditions so that the partition function is holomorphic in τ . This indeed produces the correct theta functions that we used above. In this paper we will not give further details of this calculation. A

recent discussion in which parts of this calculation were carefully worked out is [11].

5.2 Interpretation in terms of M2-branes

In the previous subsection we explained the origin of the exponential factors and the theta function. It is an interesting question whether one can also say more about the degeneracies $d_\mu(\hat{q}_0)$ with $\hat{q}_0 < 0$ that appear in the truncated characters $\chi_\mu^-(\tau)$. In particular one would like to give a more detailed accounting of those states from the point of view of string theory on $CY \times S^2 \times AdS_3$. In fact, a nice physical picture of a large class of these states was given by Gaiotto et al. in terms of M2 and anti-M2 branes which fill up Landau levels near the north and south pole of the S^2 respectively [8]. In a dilute gas approximation, this gives rise to the following contribution to the index Z :

$$Z_{\text{gas}}(\tau, y) = e^{-2\pi i c_L \tau / 24} Z_{\text{sugra}}(\tau) Z'_{GV}(\tau, \frac{1}{2}p\tau + y) Z'_{GV}(\tau, \frac{1}{2}p\tau - y) \tag{5.3}$$

where $Z_{\text{sugra}}(\tau)$ is the contribution from supergravity modes which is equal to $\prod_{n>0} (1 - e^{2\pi i n \tau})^{-n \chi_{CY}}$,⁸ and Z'_{GV} is the (reduced) Gopakumar-Vafa partition function

$$Z'_{GV}(\tau, y) = \prod_{q,m} \left[\prod_{n \geq 0} \left(1 - e^{2\pi i \tau(m+n) + 2\pi i y q} \right)^n \right]^{N_q^m} \tag{5.4}$$

where we only take into account contributions with $q \neq 0$. From the point of view of the world sheet SCFT, the M2/anti-M2 brane gas describes a collection of states that is freely generated by a collection of chiral vertex operators. It is clear that (5.3) suffers from all kinds of limitations. The dilute gas approximation will eventually break down, there could be other BPS configurations that contribute, the Landau levels can start to fill out the entire S^2 , the SU(2) quantum numbers are bounded by the level of the SU(2) current algebra, etc. Furthermore, it also does not exhibit the right behavior under spectral flow.

But suppose the above expression is correct in the low temperature regime and for small M2-brane charges. Furthermore let us focus on the regime $\hat{q}_0 < 0$ so that no black hole shall be formed, so that we are only interested in the states counted by the truncated characters $\chi_\mu^-(\tau)$. Then we can make the following identification

$$\sum_{\mu} \chi_{\mu}^{-}(\tau) e^{i\pi\tau\mu^2 + 2\pi i\mu \cdot y} \sim \left[Z_{\text{gas}}(\tau, y) \right]_{\text{trunc}} \tag{5.5}$$

where the r.h.s. is given by (5.3) but truncated in two ways: it does not only contain terms for which $\hat{q}_0 < 0$ but also the total M2-brane charge is restricted to those values for which the dilute gas approximation is valid, for which the SU(2) quantum numbers do not exceed the unitarity bounds of the CFT, etc. The most stringent interpretation of the truncation would be to consider only M2-brane charges that are restricted to lie in some fundamental domain under spectral flow, that is $q \in \Lambda^* / \Lambda + \frac{p}{2}$. However, we will leave the precise nature of the truncation deliberately vague, and leave a more precise understanding to future work.

⁸A more detailed discussion which corrects this expression can be found in [11].

In [8] it was shown that the un-truncated form of this expression together with the modularity of the elliptic genus gives a derivation of the OSV conjecture. We will now reexamine these arguments from the point of view of the Farey tail expansion, while also taking into account the spectral flow invariance. First notice that the spectral flow invariance can be restored by extracting $\chi_\mu^-(\tau)$ from this expression, and using the theta functions to build the full truncated elliptic genus. After a small calculation one obtains

$$\sum_\mu \chi_\mu^-(\tau) \theta_\mu(\tau, y) \sim e^{-2\pi i c_L \tau / 24} \sum_{k \in \Lambda} e^{i\pi \tau k^2 + 2\pi i y k} e^{\pi i p k} \left[Z_{\text{sugra}}(\tau) Z'_{GV}(\tau, \frac{1}{2} p \tau + y + k \tau) Z'_{GV}(\tau, \frac{1}{2} p \tau - y - k \tau) \right]_{\text{trunc}} \quad (5.6)$$

To obtain the Farey tail expansion of the full partition function we have to sum over the images under modular group. This is most easily done after first rewriting the r.h.s. by observing that

$$-\frac{c_L \tau}{24} + \frac{\tau}{2} k^2 + y k + \frac{y^2}{2\tau} = -\frac{1}{6\tau^2} \left((\frac{1}{2} p \tau + y + k \tau)^3 + (\frac{1}{2} p \tau - y - k \tau)^3 \right) - \frac{1}{24} c_2 \cdot \left((\frac{1}{2} p \tau + y + k \tau) + (\frac{1}{2} p \tau - y - k \tau) \right), \quad (5.7)$$

where we used on the l.h.s. d_{ab} to define the quadratic terms, while the cubic terms on the r.h.s. are defined with the help of d_{abc} . Here we recognize precisely the perturbative genus zero and genus one piece of the topological string partition function. If we therefore define

$$Z_{\text{top}}(\tau, y) = Z_{\text{sugra}}^{\frac{1}{2}}(\tau) e^{-2\pi i (\frac{1}{6\tau^2} y^3 + \frac{c_2 y}{24})} Z'_{GV}(\tau, y) \quad (5.8)$$

we can rewrite (5.6) as

$$\sum_\mu \chi_\mu^-(\tau) \theta_\mu(\tau, y) \sim e^{-\frac{\pi i y^2}{\tau}} \sum_{k \in \Lambda} e^{\pi i p k} \left[Z_{\text{top}}(\tau, \frac{1}{2} p \tau + y + k \tau) Z_{\text{top}}(\tau, \frac{1}{2} p \tau - y - k \tau) \right]_{\text{trunc}}. \quad (5.9)$$

The full partition function $Z(\tau, y)$ is equal to the sum over the modular images of (the Farey tail transform of) (5.9).

To find the entropy one has to extract the leading behavior as $\text{Im } \tau \rightarrow 0$. There is a single term which dominates, which is the $\tau \rightarrow -1/\tau$ transform of (5.9). Further, the leading behavior is determined mainly by the genus zero and genus one contributions. The subleading, exponentially suppressed "tail" contributions are a representation of the higher genus contributions. The fact that this sum is truncated is not noticeable in perturbation theory, since it hardly affects the entropy calculation. Notice that all bounds on the M2-brane charges disappear for large M5 charge p^a , and hence in this regime we can ignore the truncation all together.

So we can approximate the full partition function by its leading term in the Farey tail expansion

$$Z(\tau, y) \sim \sum_{k \in \Lambda} e^{\pi i p k} Z_{\text{top}} \left(-\frac{1}{\tau}, \frac{-\frac{1}{2} p + y - k}{\tau} \right) Z_{\text{top}} \left(-\frac{1}{\tau}, -\frac{-\frac{1}{2} p - y + k}{\tau} \right). \quad (5.10)$$

where we removed the truncation, ignored some powers of τ and used the fact that the prefactor in (5.9) precisely cancels under the $\tau \rightarrow -1/\tau$ transformation.

To get from here to the OSV conjecture for the entropy it is even nicer to keep the sum over d in the Farey Tail expansion as well, which results in

$$\begin{aligned}
 Z(\tau, y) \sim & \sum_{d \in \mathbb{Z}} \sum_{k \in \Lambda} e^{\pi i p k} Z_{\text{top}} \left(-\frac{1}{\tau + d}, \frac{-\frac{1}{2}p + y - k}{\tau + d} \right) \\
 & \times Z_{\text{top}} \left(-\frac{1}{\tau + d}, \frac{-\frac{1}{2}p - y + k}{\tau + d} \right)
 \end{aligned} \tag{5.11}$$

The entropy $\Omega(p, q)$ can now be written as a multiple contour integrals of

$$Z(\tau, y) e^{-2\pi i (q_0 \tau + q_a y^a)}. \tag{5.12}$$

Crucially, this is a properly periodic function of y . The peculiar phase $e^{\pi i p k}$ in (5.11) cancels against a similar phase coming from $e^{-2\pi i (q_0 \tau + q_a y^a)}$, because the membrane charge lattice is shifted by $p/2$. Therefore, the contour integrals together with the sum over $k \in \Lambda$ and the sum over d can be rewritten as integrals over the entire imaginary axis, so that at the end of the day the entropy becomes an inverse Laplace transform of $|Z_{\text{top}}|^2$, which is precisely the OSV conjecture.

Notice that we have been somewhat inaccurate in keeping track of the $p/2$ shift in part of the above computation, as on the right hand side of (5.6) the membrane charge lattice is centered around zero rather than $p/2$. The correct expression involves a sum $\sum_{k \in \Lambda + p/2}$ rather than $\sum_{k \in \Lambda}$. This has little effect, as one can at the end, in (5.11), undo this shift by a shift of y , which in turn generates an extra phase $e^{\pi i p q}$ in (5.12). This phase should indeed be present, as the definition (3.14) of the generalized elliptic genus did precisely involve such an extra phase.

It is worth emphasizing that the occurrence of the topological string in the degeneracies of the M2 and anti-M2 brane gas (à la Gopakumar-Vafa) and the OSV-conjecture are naturally related by the Farey tail expansion. In fact, each term in the Farey tail expansion has a representation in terms of a square of a topological string partition function with coupling constant equal to $(a\tau + b)/(c\tau + d)$. This representation is however approximate because it assumes a complete decoupling of the M2 and anti-M2 brane states, which clearly breaks down for large M2-charges or for small M5-brane charge. We expect that a more complete analysis will involve corrections in a way similar to the ones found for 2d Yang-Mills [21]. We hope to come back to this point in a future paper.

6. Discussion

In this paper we present the Farey tail expansion for $N = 2$ attractor black holes. The central idea of this expansion is that the one has to first truncate the partition function so that it includes only particular low energy states, and then sum over all images of it under the modular group. Each term can be interpreted as representing the contribution of a particular (semi-)classical background. The formula can thus be regarded as partly

microscopic (as the states counted in the "tail") as well as macroscopic (as the sum over classical backgrounds). We would like to emphasize that in this expansion, there is no one to one correspondence between microstates and gravitational backgrounds (as suggested by Mathur et al, see e.g. [22]). Quite on the contrary, the major part of the entropy is carried by one particular black hole background.

The supergravity interpretation of the Farey expansion involves a natural complete collection of backgrounds of a given type. It is natural to ask whether the expansion can be refined by including more general macroscopic backgrounds. It is indeed likely that such refinements exist, but one expects that these will follow a similar pattern: one has to truncate the microscopic spectrum even further and replace the contribution of the omitted states by certain classical backgrounds. Here one can think of various type of backgrounds, such as multi-centered solutions ("baby universes"), bubbling solutions that deform the horizon geometry, black rings... etc. A large class of such solutions is known, but the list is presumably incomplete, and it remains an interesting problem to use them in a systematic or let alone exact manner.

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A. Restoring the $\bar{\tau}$ -dependence

In our discussion of the elliptic genus, from equation (3.21) onwards we put $\tau = \bar{\tau}$ since we argued that the $\bar{\tau}$ -dependence can always be reconstructed by using (3.18) and thus we do not need an extra potential $\bar{\tau}$ to label the charges of the supersymmetric states. This argument, though, is only formal in the sense that we need the $\bar{\tau}$ -dependence for the sum to converge. For completeness, we present the computation with $\bar{\tau}$ -dependence restored in this appendix.

We begin with the counterpart of (3.21) but now with proper $\bar{\tau}$ dependence and also the $e^{\pi i p \cdot q}$ included

$$Z_{p^a}(\tau, \bar{\tau}, y_L, y_R) = \sum_{\{q_0, q \in \Lambda^* + p/2\}} d(q_0, q) e^{\pi i p \cdot q} e^{2\pi i \tau q_0} e^{i\pi(\tau - \bar{\tau})q_R^2} e^{2\pi i y \cdot q}. \quad (\text{A.1})$$

The spectral flow argument for the BPS states implies again

$$d\left(n + \frac{1}{2}q_L^2, q_L\right) = d_\mu(n), \quad (\text{A.2})$$

with $q = \mu \bmod \Lambda$, $\mu \in \Lambda^*/\Lambda + p/2$. Therefore we can decompose the elliptic genus in holomorphic modular forms and non-holomorphic theta functions:

$$Z_{p^a}(\tau, \bar{\tau}, y_L, y_R) = \sum_{\mu \in \Lambda^*/\Lambda + \frac{p}{2}} \chi_\mu(\tau) \theta_\mu(\tau, \bar{\tau}, y^a), \quad (\text{A.3})$$

with

$$\chi_\mu(\tau) = \sum_{-\frac{c_L}{24} \leq n} d_\mu(n) e^{2\pi i \tau n}, \quad (\text{A.4})$$

$$\begin{aligned} \theta_\mu(\tau, \bar{\tau}, y_L, y_R) &= \sum_{k \in \Lambda} e^{\pi i p \cdot \mu} e^{\pi i p \cdot k} e^{2\pi i \tau \frac{(\mu_L + k_L)^2}{2}} e^{-2\pi i \bar{\tau} \frac{(\mu_R + k_R)^2}{2}} \\ &\times e^{2\pi i y_L \cdot (\mu_L + k_L)} e^{2\pi i y_R \cdot (\mu_R + k_R)} \end{aligned} \quad (\text{A.5})$$

The transformation properties of these theta-functions can be calculated straight forwardly [23]. First of all they satisfy

$$\begin{aligned} \theta_\mu(\tau + 1, \bar{\tau} + 1, y_L, y_R) &= \sum_{k \in \Lambda} e^{\pi i p \cdot \mu} e^{\pi i p \cdot k} e^{\pi i (\mu + k)^2} e^{2\pi i \tau \frac{(\mu_L + k_L)^2}{2}} e^{-2\pi i \bar{\tau} \frac{(\mu_R + k_R)^2}{2}} \\ &\times e^{2\pi i y_L \cdot (\mu_L + k_L)} e^{2\pi i y_R \cdot (\mu_R + k_R)} \end{aligned} \quad (\text{A.6})$$

$$= e^{\pi i \mu^2} \theta_\mu(\tau, \bar{\tau}, y_L, y_R). \quad (\text{A.7})$$

The presence of the Freed-Witten anomaly, that is, the fact that $\mu' \equiv \mu - \frac{p}{2} \in \Lambda^*/\Lambda$, is essential for the above equation to hold: since $k \in \Lambda$, we can think of the M2 brane represented by k as wrapping an integral two-cycle embedded inside the divisor \mathcal{P} . In this case one can apply the adjunction formula

$$Q \cdot Q + Q \cdot [\mathcal{P}] = 2g - 2 \quad \Rightarrow \quad k^2 = k \cdot p \pmod{2}, \quad (\text{A.8})$$

and exploit the relation

$$\begin{aligned} (\mu + k)^2 &= k^2 + 2 \left(\mu' + \frac{p}{2} \right) \cdot k + \left(\mu' + \frac{p}{2} \right)^2 \\ &= k^2 + k \cdot p + 2\mu' \cdot k + \left(\mu' + \frac{p}{2} \right)^2 \\ &= \mu^2 \pmod{2} \end{aligned} \quad (\text{A.9})$$

to arrive at the desired transformation property of the theta-function.

Under S-transformation, one has

$$\begin{aligned} \theta_\mu\left(\frac{-1}{\tau}, \frac{-1}{\bar{\tau}}, \frac{y_L}{\tau}, \frac{y_R}{\bar{\tau}}\right) &= \sum_{k \in \Lambda} e^{-\pi i p \cdot \mu} e^{-\pi i p \cdot k} e^{-\frac{2\pi i}{\tau} \frac{(\mu_L + k_L)^2}{2}} e^{\frac{2\pi i}{\bar{\tau}} \frac{(\mu_R + k_R)^2}{2}} \\ &\times e^{\frac{2\pi i y_L}{\tau} \cdot (\mu_L + k_L)} e^{\frac{2\pi i y_R}{\bar{\tau}} \cdot (\mu_R + k_R)} \\ &= \sqrt{\frac{|\Lambda|}{|\Lambda^*|}} e^{-\pi i p^2/2} e^{\pi i \left(\frac{y_L^2}{\tau} - \frac{y_R^2}{\bar{\tau}} \right)} (-i\tau)^{(b_2-1)/2} (i\bar{\tau})^{1/2} \times \\ &\sum_{\delta \in \Lambda^*/\Lambda} e^{-2\pi i \mu \cdot \delta} \theta_\delta(\tau, \bar{\tau}, y_L, y_R), \end{aligned} \quad (\text{A.10})$$

where we have performed a Poisson resummation. We can see that the theta-functions have weights $(\frac{b_2-1}{2}, \frac{1}{2})$ and from this we can deduce that the weights of the characters $\chi_\mu(\tau)$ are $(\frac{-b_2}{2} - 1, 0)$.

Notice that all the $\bar{\tau}$ -dependence in $Z_{p^a}(\tau, \bar{\tau}, y_L, y_R)$ is contained in the theta function factor as before; in other words, the incorporation of the $\bar{\tau}$ -dependence does not change the fact that all the information about the black hole degeneracies is contained in holomorphic characters $\chi_\mu(\tau)$. This is of course again a consequence of the supersymmetry condition. The desired transformation properties of the elliptic genus require that the vector valued modular form $\chi_\mu(\tau)$ transforms in a conjugate representation as the theta function. Its transformation properties are

$$\begin{aligned} \chi_\mu(\tau + 1) &= e^{-\pi i \mu^2} \chi_\mu(\tau), \\ \chi_\mu\left(\frac{-1}{\tau}\right) &= \sqrt{\frac{|\Lambda^*|}{|\Lambda|}} e^{\pi i p^2/2} e^{\pi i (b_2-2)/4} \left(\frac{1}{\tau}\right)^{(b_2+2)/2} \sum_{\delta \in \Lambda^*/\Lambda} e^{2\pi i \mu \cdot \delta} h_\delta(\tau). \end{aligned} \quad (\text{A.11})$$

In order to write the elliptic genus as a Rademacher expansion, we have to perform a Farey transform to $\chi_\mu(\tau)$. The transformed elliptic genus $\tilde{Z}_{p^a}(\tau, \bar{\tau}, y_L, y_R)$ has positive weights $\tilde{w} = \frac{5}{2} + b_2$, $\tilde{\bar{w}} = \frac{1}{2}$. $\tilde{Z}_{p^a}(\tau, \bar{\tau}, y_L, y_R)$ written as a Rademacher expansion reads

$$\begin{aligned} \tilde{Z}_{p^a}(\tau, \bar{\tau}, y_L, y_R) &= \sum_{\mu \in \Lambda^*/\Lambda + \frac{p}{2}} \sum_{\gamma \in \Gamma/\Gamma_\infty} \tilde{\chi}_\mu^- \left(\frac{a\tau + b}{c\tau + d} \right) \theta_\mu \left(\frac{a\tau + b}{c\tau + d}, \frac{a\bar{\tau} + b}{c\bar{\tau} + d}, \frac{y_L}{c\tau + d}, \frac{y_R}{c\bar{\tau} + d} \right) \\ &\quad (c\tau + d)^{-(\frac{5}{2} + b_2)} (c\bar{\tau} + d)^{-\frac{1}{2}} e^{-\pi i \frac{c y_L^2}{c\tau + d}} e^{\pi i \frac{c y_R^2}{c\bar{\tau} + d}}, \end{aligned} \quad (\text{A.12})$$

where

$$\tilde{\chi}_\mu^-(\tau) = \sum_{-\frac{cL}{24} \leq n < 0} \tilde{d}_\mu(n) e^{2\pi i \tau n} \quad (\text{A.13})$$

now contains only the "polar part" as before. This is our convergent attractor Farey Tail with the explicit $\bar{\tau}$ -dependence.

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Erratum

Our paper [24] relied on the properties of ref. [25] to rewrite the “Fareytail” transform of the $\mathcal{N} = (0, 4)$ elliptic genus as a Poincaré series. Don Zagier has noticed that some properties of the Fareytail transform, originally assumed in ref. [25], are erroneous. In this erratum, we point out where [24] relies on the erroneous assumptions of [25]. We refer to ref. [26] for more details of the Fareytail transform.

The physical interpretation of the Poincaré series, described in section 5 of [24], remains unchanged. Ref. [26] presents a method to write the elliptic genus directly as a (regularized) Poincaré series, circumventing the problems of the Fareytail transform.

The Fareytail transform is given by eq. (4.11) in section 4.2 of [24]. This section explains that the weight w_χ is given by $-1 - r/2$. Mimicking [25], we assumed that the function $\tilde{\chi}_\mu(\tau)$ is a proper (vector-valued) modular form for general $w_\chi \leq 0$, including half-integer values. This assumption is mistaken, $\tilde{\chi}_\mu(\tau)$ transforms only in a covariant way for integral values of w_χ , or even r . The appendix of [24] restores the $\bar{\tau}$ -dependence and applies also the Fareytail transform. In that case, the Fareytail transform only leads to a proper modular form in case b_2 is even. We conclude that the Fareytail transform of the elliptic genus of an M-theory black hole only maps to a modular form in case the second Betti number of the corresponding Calabi-Yau is even. The Poincaré series (4.12) and (A.12) are in this case non-vanishing functions and transform covariantly under modular transformations. The series in eqs. (4.12) and (A.12) in fact vanish when r and/or b_2 are odd.

Added references

- [24] J. de Boer, M.C.N. Cheng, R. Dijkgraaf, J. Manschot and E. Verlinde, *A farey tail for attractor black holes*, *JHEP* **11** (2006) 024 [[hep-th/0608059](#)].
- [25] R. Dijkgraaf, J.M. Maldacena, G.W. Moore and E.P. Verlinde, *A black hole farey tail*, [hep-th/0005003](#).
- [26] J. Manschot and G.W. Moore, *A modern farey tail*, [arXiv:0712.0573](#).